The Variance of the Fibonacci Partition Function

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Fibonacci Partitions

For any $n \in \mathbb{N}$, $R(n)$ is defined to be the number of solutions to the equation

$$F_{m_1} + F_{m_2} + \cdots + F_{m_r} = n \qquad (1)$$

where F_m represents the m^{th} Fibonacci number and $m_r > \cdots > m_2 > m_1 \geq 2$ are integers. In other words, $R(n)$ is the number of ways to *partition* n into distinct Fibonacci numbers. The function R is thus called the Fibonacci partition function.

The Erratic Nature of R

The behaviour of the Fibonacci partition function is erratic. For example, $R(F_m - 1) = 1$ for any positive integer m. However, as $m \to \infty$, $R(F_m^2 - 1) \to \infty$. In fact, $R(F_m^2 - 1) = F_m$.

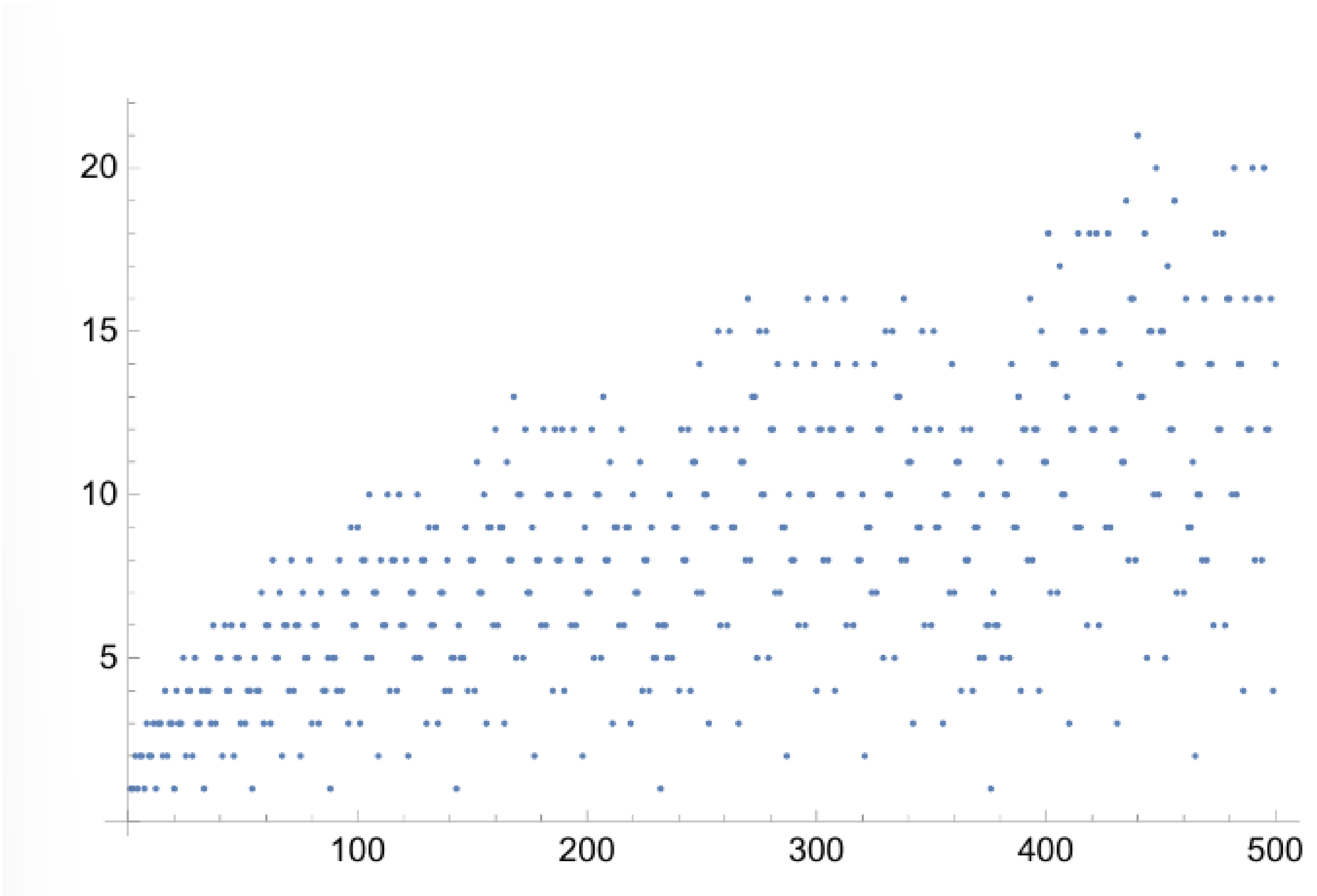

Figure 1: Number of Fibonacci partitions for n against n

Finding The Variance

To quantify this spread of values taken by R, we try to understand the quantity $\frac{V(n)}{n+1} - \frac{A(n)^2}{(n+1)^2}$, where

$$V(n) = R(0)^2 + \cdots + R(n)^2 \text{ and}$$
$$A(n) = R(0) + \cdots + R(n).$$

This quantity can thus be termed the *variance* of R. It was shown in [1] that $A(n) \asymp n^{\lambda}$ where $\lambda = \frac{\log 2}{\log \phi} \approx 1.44$ and $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Hence, we only need study the function V.

Notice that we can interpret $R(n)^2$ as the number of 'pairs' of solutions to the equation (1). That is, the number of solutions to

$$F_{p_1} + F_{p_2} + \cdots + F_{p_s} = F_{m_1} + F_{m_2} + \cdots + F_{m_r} = n.$$

As a result, we can view $V(n)$ as the number of solutions to

$$F_{p_1} + F_{p_2} + \cdots + F_{p_s} = F_{m_1} + F_{m_2} + \cdots + F_{m_r} \leq n.$$

Case Analysis

We now consider $V(F_m) - V(F_{m-1})$, which is the number of solutions to

$$F_{m-1} < F_{p_1} + F_{p_2} + \cdots + F_{p_s} = F_{m_1} + F_{m_2} + \cdots + F_{m_r} \leq F_m.$$

Clearly, m_r and p_s cannot be greater than m. Moreover, the identity

$$F_2 + F_3 + \cdots + F_{m-3} = F_{m-1} - 2 < F_{m-1}$$

means m_r and p_s must be at least $m-2$. In addition, $\{m_r, p_s\} \neq \{m, m-2\}$ since the above identity implies

$$F_2 + F_3 + \cdots + F_{m-2} < F_m.$$

We consequently have five possibilities for $\{m_r, p_s\}$:

1. $\{F_m\}$
2. $\{F_{m-1}\}$
3. $\{F_{m-2}\}$
4. $\{F_m, F_{m-1}\}$
5. $\{F_{m-1}, F_{m-2}\}$

A Recursive Formula

Each of our five cases reduces the original inequality to a familiar equation/inequality but for a smaller value of m. This is in part due to the fact that $F_m = F_{m-1} + F_{m-2}$. For example, case 2. gives us the inequality

$$0 < F_{p_1} + \cdots + F_{p_{s-1}} = F_{m_1} + \cdots + F_{m_{r-1}} \leq F_{m-2}$$

which has $V(F_{m-2}) - V(0)$ solutions. Adding solutions for each case in this way gives the recursion

$$V_m = 2V_{m-1} + 3V_{m-2} - 4V_{m-3} - 2V_{m-4} + 2V_{m-5}$$
$$-2R_{m-1} - 2R_{m-4} + 2R_{m-5} + 1$$

where $V_m = V(F_m)$ and $R_m = R(F_m)$.

Now, the formula $R_m = \lfloor m/2 \rfloor$, deduced in [2], can be used to prove that the recursion is solved by

$$V_m = \sum_{i=1}^{5} c_i \lambda_i^m + m \left\lfloor \frac{m}{2} \right\rfloor - \frac{m^2}{4}$$

where the c_i are fixed real numbers chosen to fit initial conditions and the λ_i are the (real, distinct) roots of the polynomial

$$x^5 - 2x^4 - 3x^3 + 4x^2 + 2x - 2.$$

Letting $\lambda_1 \approx 2.48$ be the (unique) root of largest magnitude, we see that $V_m \sim c_1 \lambda_1^m$ with $c_1 \approx 0.0735$. Finally, it can be shown that $V(n) \asymp n^p$, with $p = \frac{\log(\lambda_1)}{\log(\phi)} \approx 1.89$, in a similar way to the method in [1].

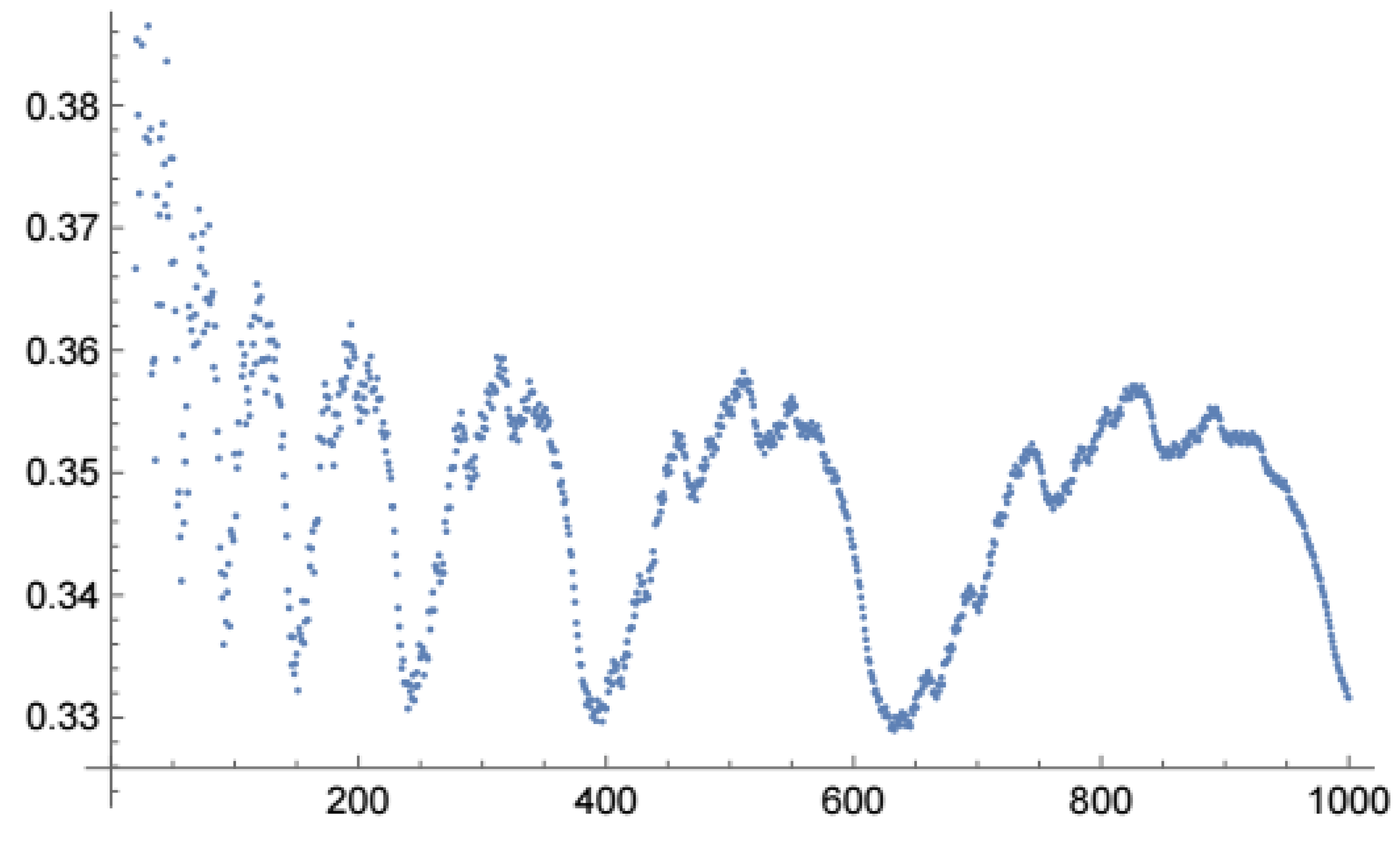

Figure 2: $\frac{V(n)}{n^p}$ against n

References

[1] S. Chow and T. Slattery, On Fibonacci partitions, J. Number Theory **225** (2021), 310–326.

[2] L. Carlitz, Fibonacci representations, Fibonacci Quart. **6** (1968), 193–220.

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