# The Effects of Viscosity and Heat Conduction On Sound Waves

## Introduction

- The effects of viscosity and heat conduction are not typically considered.
- Consider the damping of the sound waves by analyzing **dispersion relation**.
- Derive the dispersion relation and the damping rate for a non-ideal compressible fluid.
- Heating of the fluid due to viscosity.

## Governing equation

### Non-ideal fluid case:

✓ Navier-Stokes equation:

$$\rho\left(\frac{\partial \bar{u}}{\partial t}+\bar{u}\cdot\nabla\bar{u}\right)=-\nabla p+\mu\nabla^2\bar{u}+\left(\zeta+\frac{1}{3}\mu\right)\nabla(\nabla\cdot\bar{u})$$

✓ Continuity equation:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\bar{u})=0$$

✓ Energy equation:

$$\rho C_v\left(\frac{\partial T}{\partial t}+\bar{u}\cdot\nabla T+(\gamma-1)T\nabla\cdot\bar{u}\right)=\mu\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right)+\left(\zeta+\frac{1}{3}\mu\right)(\nabla\cdot\bar{u})^2+K\nabla^2 T$$

✓ Ideal gas law:

$$p=\frac{k_B}{\bar{m}}\rho T$$

### Ideal fluid case:

✓ Euler's equation

$$\rho\left(\frac{\partial \bar{u}}{\partial t}+\bar{u}\cdot\nabla\bar{u}\right)=-\nabla p$$

✓ Continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\bar{u})=0$$

✓ Adiabatic energy equation

$$\left(\frac{\partial}{\partial t}+\bar{u}\cdot\nabla\right)\left(\frac{P}{\rho^{\gamma}}\right)=0$$

Wave equation;

$$\frac{\partial^2 p_1}{\partial t^2}=c_s^2\nabla^2 P\text{, where } c_s=\sqrt{\frac{\gamma P_0}{\rho_0}}$$

## Mathematical method

$\omega=\pm C_s k$

Linear theory of sound waves:

$$p=p_0+p_1(\bar{r},t)$$

Static term and dynamic term where the dynamic term can be treated as first order small term

Complex exponential method:

$$p_1(\bar{r},t)=\mathrm{Re}\{\hat{p}_1\,exp(\mathrm{i}(\bar{k}\cdot\bar{r}-\omega t))\}$$

Matrix formation:

$$\begin{bmatrix}\rho_0 & -\gamma p_0 & 0 & 0 & 0\\ 0 & -\omega & \rho_0 k_x & \rho_0 k_y & \rho_0 k_z\\ k_x & 0 & -\rho_0\omega & 0 & 0\\ k_y & 0 & 0 & -\rho_0\omega & 0\\ k_z & 0 & 0 & 0 & -\rho_0\omega\end{bmatrix}\begin{bmatrix}\hat{p}_1\\ \hat{\rho}_1\\ u_x\\ u_y\\ u_z\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

Fourier components – focus on a given wavenumber & frequency (linear theory)

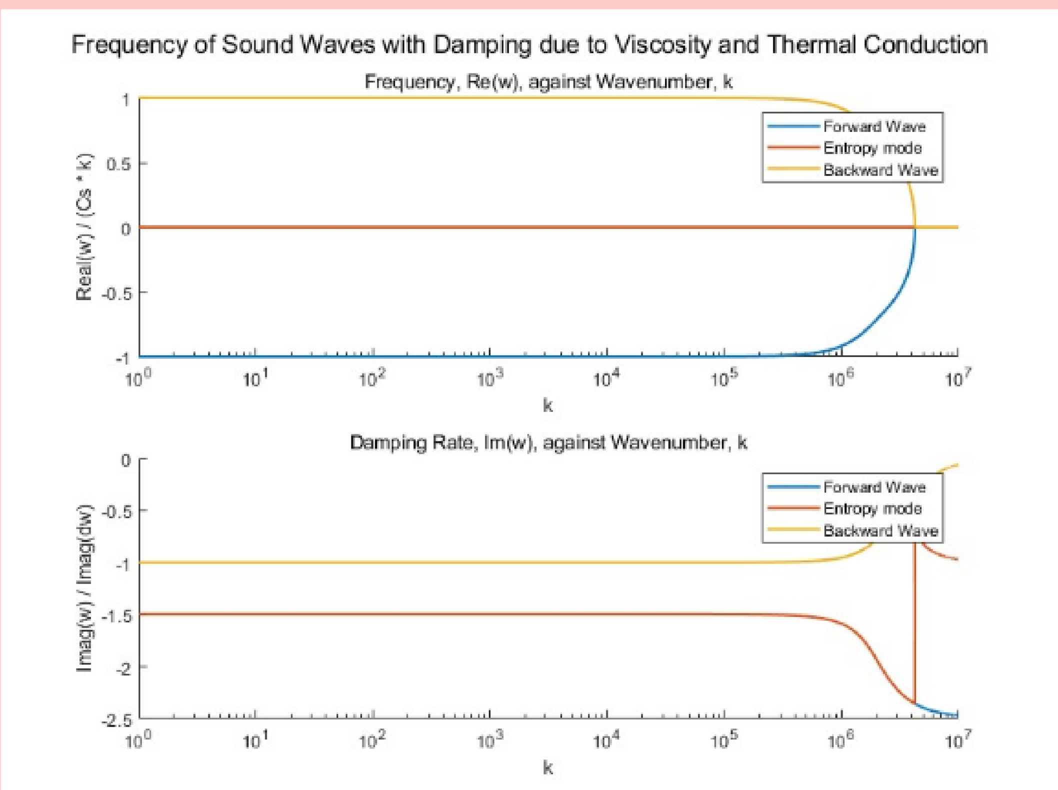
$$T(x,t)=T_h(x,t)+T_p(x,t)$$

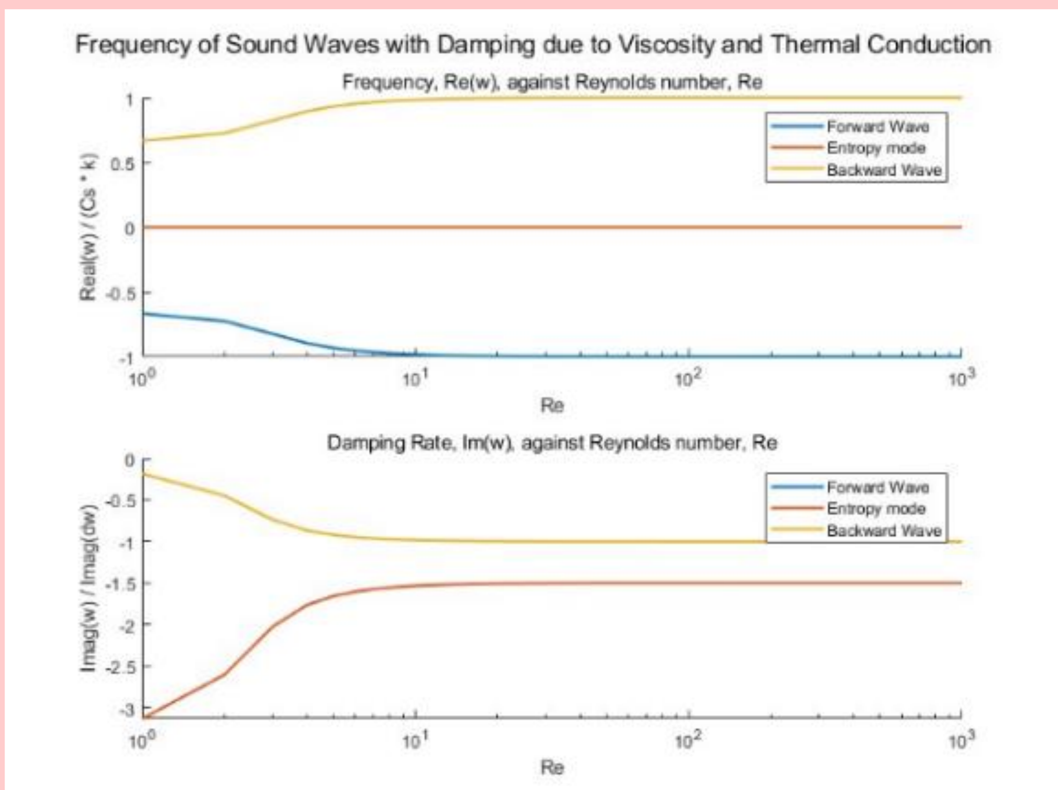
$$T_h(x,t)=\int_{-\infty}^{\infty}\left(\frac{D}{\pi t}\right)^{\frac{1}{2}}\cdot e^{\frac{-(x-y)^2 D}{t}}\cdot A\cos(ky)dy$$

$$T_p(x,t)=\int_0^t\int_{-\infty}^{\infty}\left(\frac{D}{\pi(t-\tau)}\right)^{\frac{1}{2}}\cdot e^{\frac{-(x-y)^2 D}{t-\tau}}\cdot\{B\sin(ky-\omega t)+C[\sin(ky-\omega t)]^2\}dyd\tau$$

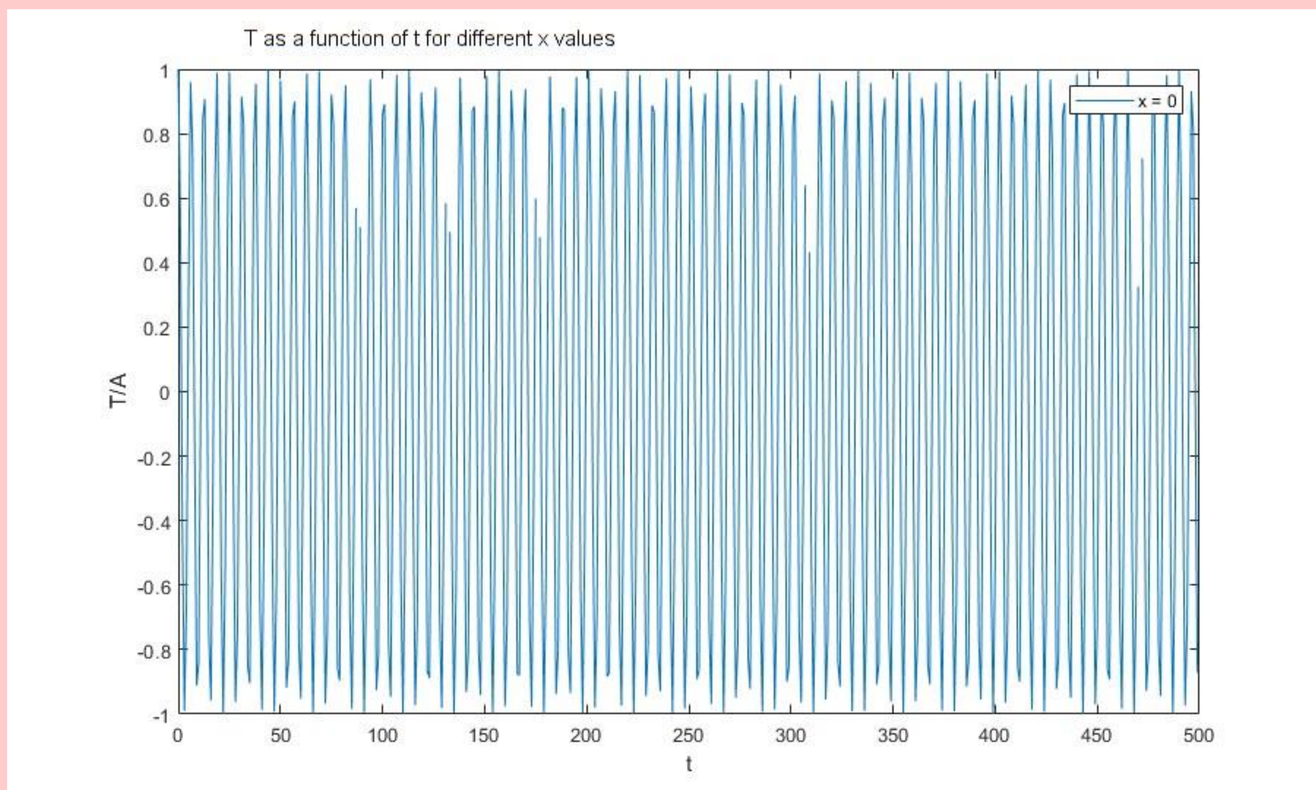
**The temperature function (solved by using Green's function)**

## Numerical results

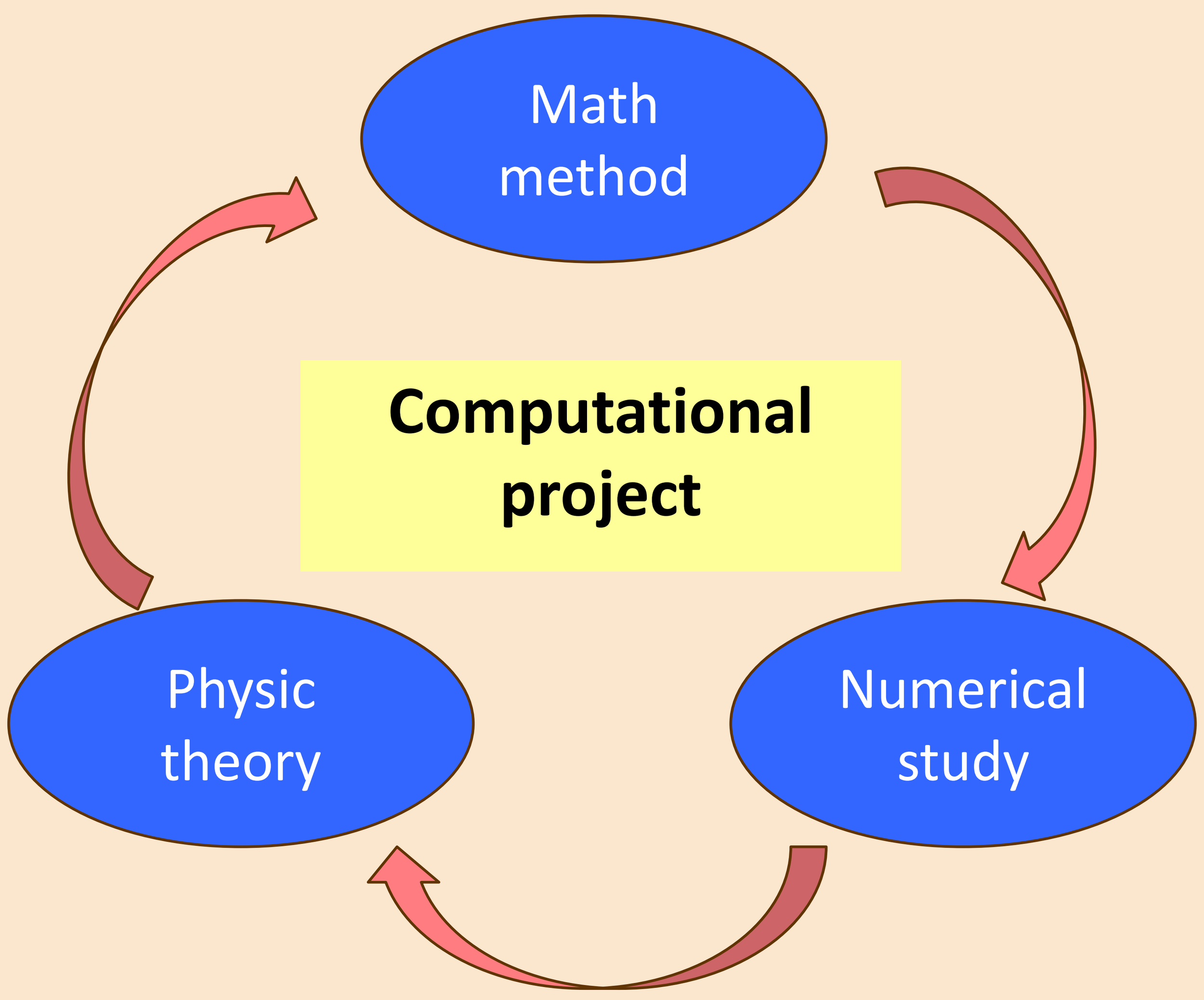



Frequency change for Helium with damping rate with respect of k (left) and Re (right)




Temperature change for air with wave compression and expansion




## Summary

**What we have done:**

- Studied the effect of viscosity and heat conduction – dispersion relation
- Heating of gas due viscosity – Green's function solution
- Numerical results

**Future investigation:**

- Investigate other scenarios where heating could occur e.g. industrial or astrophysical conditions
- Full coupled problem of viscous heating – changes to temperature influence the velocity field
- Reduce numerical error in code

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SURE – Sheffield Undergraduate Research Experience


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